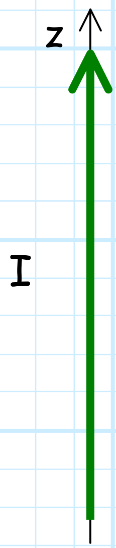
Example: The Uniform, Infinite Line of Current

Consider electric current I flowing along the **z-axis** from $z=-\infty$ to $z=\infty$. What magnetic flux potential $\mathbf{B}(\bar{r})$ is created by this current?



$$\overline{d\ell} = \hat{a}_z \, dz'$$

$$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$$
$$= \rho\cos\phi\,\hat{a}_x + \rho\sin\phi\,\hat{a}_y + z\,\hat{a}_z$$

$$\bar{r}' = z'\,\hat{a}_z \qquad (x'=0, y'=0)$$

$$|\bar{r}-\bar{r}'| = \sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi + (z-z')^2}$$
$$= \sqrt{\rho^2 + (z-z')^2}$$

We can determine the magnetic flux density by applying the **Biot-Savart Law**:

$$
\begin{aligned}
\mathbf{B}(\bar{\mathbf{r}}) &= \frac{\mu_0 I}{4\pi} \oint_C \frac{\overline{d\ell'} \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3} \\
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\hat{a}_z \times \left[\rho \cos\phi\, \hat{a}_x + \rho \sin\phi\, \hat{a}_y + (z - z')\, \hat{a}_z\right]}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\, dz' \\
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho \cos\phi\, \hat{a}_y - \rho \sin\phi\, \hat{a}_x}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\, dz' \\
&= \frac{\mu_0 I}{4\pi} \left(\rho \cos\phi\, \hat{a}_y - \rho \sin\phi\, \hat{a}_x\right) \int_{-\infty}^{\infty} \frac{du}{\left[\rho^2 + u^2\right]^{3/2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\rho\, \hat{a}_\phi\right) \Bigg|_{-\infty}^{\infty} \frac{u}{\rho^2 \sqrt{\rho^2 + u^2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\rho\, \hat{a}_\phi\right) \frac{2}{\rho^2} \\
&= \frac{\mu_0 I}{2\pi \rho}\, \hat{a}_\phi
\end{aligned}
$$

Therefore, the magnetic flux density **created** by a "wire" with current I flowing along the z-axis is:

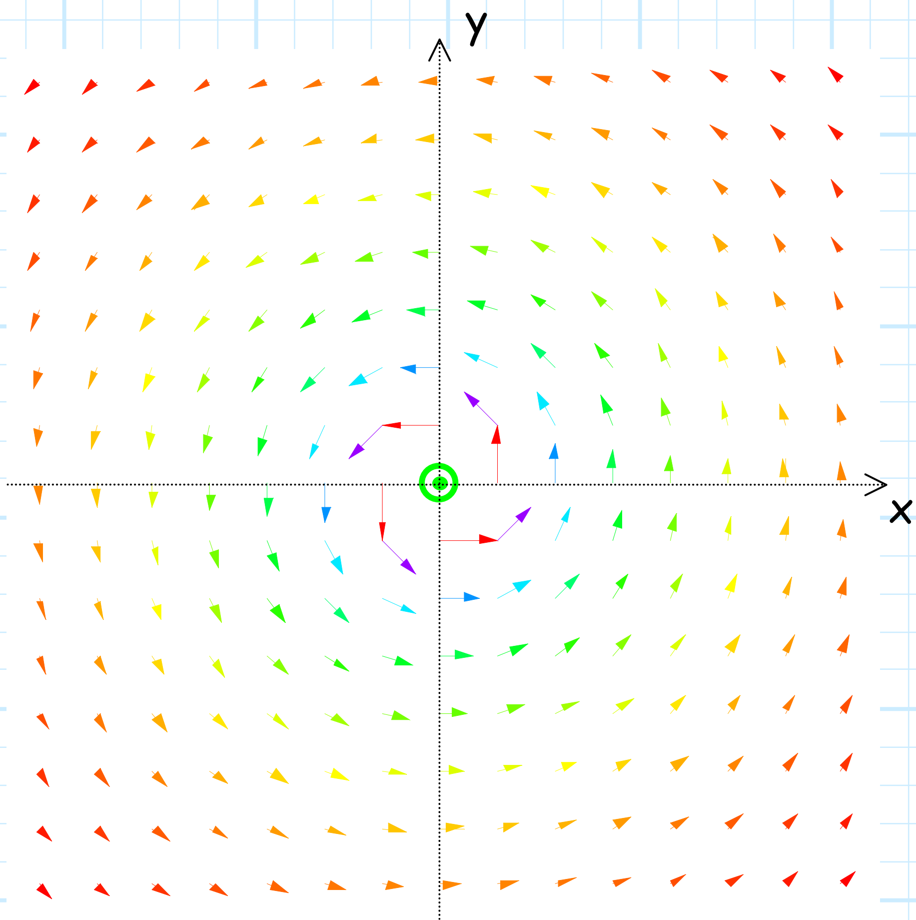
$$
\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0 I}{2\pi \rho}\, \hat{a}_\phi
$$

Think about what this expression tells us about magnetic flux density:

* The magnitude of $\mathbf{B}(\bar{r})$ is proportional to $1/\rho$, therefore magnetic flux density **diminishes** as we move farther from "wire".

* The direction of $\mathbf{B}(\bar{r})$ is $\hat{a}_\phi$. In other words, the magnetic flux density points in the direction **around** the wire.

Plot of vector field $\mathbf{B}(\bar{r})$ on the x-y plane, resulting from current I flowing along the z-axis



= current I flowing out of this page.

Or, plotting in 3-D: